Noname manuscript No.
(will be inserted by the editor)

Offbeat Questions in L^1-approximation

Dmitry Khavinson

Received: date / Accepted: date

1 Introduction

Let $\mathbb{D} := \{|z| < 1\}$, $\mathbb{T} = \partial\mathbb{D}$ denote the unit disk and circle respectively. $A(\mathbb{D}) := \left\{f \in C\left(\overline{\mathbb{D}}\right) \cap \mathrm{Hol}(\mathbb{D})\right\}$ stands for the "disk-algebra" of functions analytic in $\mathbb{D}$ and continuous in the closed disk. For $0 < p < \infty$,

$$H^p = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^p}^p := \sup_{r<1} \left\{ \int\limits_{T} \left| f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} \right\} < \infty \right\}$$

denotes the Hardy space. As usual, $H^\infty = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_\infty := \sup\limits_{z\in\mathbb{D}} |f(z)| < \infty \right\}$,

$$A^p(\mathbb{D}) := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{A^p}^p := \int\limits_{\mathbb{D}} |f|^p \frac{dA}{\pi} < \infty \right\}$$

denotes the Bergman space — cf. [1,2,5,6], ($d\theta$, dA are, of course, Lebesgue measures on $\mathbb{T}, \mathbb{D}$, respectively).

Consider the following trivial question:

Let $f \in C(\mathbb{T})$, $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{T}\} = 1$, and suppose $\exists \{f_n\} \in H^1$, $f_n \to f$ in $L^1\left(\mathbb{T}; \frac{d\theta}{2\pi}\right)$, i.e., $\lim\limits_{n\to\infty} \int_{\mathbb{T}} |f - f_n| \frac{d\theta}{2\pi} = 0$.

Can we modify the sequence $\{f_n\}$ and find $\{g_n\} \in H^\infty$, $\|g_n\|_\infty \le 1$ such that still $g_n \to f$ in L^1? The answer is, of course, "yes". Indeed, by the F. & M. Riesz theorem, $f \in A(\mathbb{D})$, so $g_r := f(rz) \to f$ (uniformly in as $r \to 1$) and, clearly, by Lebesgue Bounded Convergence Theorem, $g_r \to f$ in L^1 while $\|g_r\|_\infty \le 1$. (Even simpler, just take $g_n = f$, $n = 1, 2, \ldots$.) The following highly nontrivial and useful extension to the context of uniform algebras is the celebrated Hoffman–Wermer theorem. (For all

Dmitry Khavinson
Department of Mathematics & Statistics, University of South Florida, Tampa, FL 33620-5700
E-mail: dkhavins@usf.edu

basic facts and terminology concerning the theory of uniform algebras we refer to [4].)

Let Γ be a compact metric space, $C(\Gamma)$ denotes the Banach algebra of all continuous functions on Γ. Let A be a uniform algebra on Γ, $A \subset C(\Gamma)$. As usual, M_A denotes the maximal ideal space of the algebra A. For any complex homeomorphism φ of $A(\varphi \in M_A)$, M_φ denotes the set of all representing measures μ of φ.

The following result is due to Hoffman and Wermer [4, Ch. 11, Thm. 7.2].

Theorem 1 *Let A and $\varphi \in M_A$ be as above. Let $f \in C(\Gamma)$ and assume that f belongs to the closure of A in $L^1(\mu)$ for all $\mu \in M_\varphi$. Then, $\exists \{f_n\} : f_n \in A$, such that $\|f_n\|_{C(\Gamma)} \leq \|f\|_{C(\Gamma)}$ and $f_n \to f, \mu$ — a.e., for all $\mu \in M_\varphi$.*

The following question was initially posed to the author by J. Wermer in 1980. (We shall refer to it as an ε-version of Thm. 1.)

Suppose $f \in C(\Gamma)$ can be approximated within ε by $H^1(\mu) := \left\{L^1(\mu)\text{-closure of } A, \mu \in M_\varphi\right\}$, i.e., $\mathrm{dist}_{L^1(\mu)}\left(f, H^1(\mu)\right) \leq \varepsilon$.

Question 1 Does there exist $g \in A$, $\|g\|_\infty := \|g\|_{C(\Gamma)} \leq \|f\|_{C(\Gamma)}$, such that $\mathrm{dist}_{L^1(\mu)}(f,g) \leq C\varepsilon$, where the constant C only depends on φ?

Equivalently, if we denote by $B_A = \left\{f \in A : \|f\|_{C(\Gamma)} \leq 1\right\}$ the unit ball of A and $f \in C(\Gamma)$, $\|f\|_\infty = 1$ and $\mathrm{dist}_{L^1(\mu)}(f,A) \leq \varepsilon$, the question is whether the $\mathrm{dist}_{L^1(\mu)}(f,B_A) = O(\varepsilon)$, where the constant in $O(\varepsilon)$ only depends on φ.

As stated, the answer, in general, is "No".

More precisely, in '81, DK (unpublished) proved that for the Dirichlet algebras (cf. [4]), i.e., for such $A \subset C(\Gamma)$ that $\overline{\mathrm{Re}\, A} = C_{\mathbb{R}}(\Gamma)$, the answer is "Yes" if we replace $O(\varepsilon)$ by the worse estimate $O\left(\sqrt[4]{\varepsilon}\right)$.

In [11], the answer "Yes" was extended to hypodirichlet algebas, also with the estimate $O\left(\sqrt[4]{\varepsilon}\right)$. (Recall that A is called a hypodirichlet algebra if $\overline{\mathrm{Re}\, A}$ has finite codimension in $C_{\mathbb{R}}(\Gamma)$ — [4].)

Moreover, in the most basic case of a Dirichlet algebra, when $A := A(\mathbb{D})$ (the disk algebra), it was shown in [11] that even in the case of the disk-algebra in the unit disk one cannot improve the estimate to $O(\varepsilon)$, thus giving a negative answer to Wermer's question. Surprisingly, the latter result as indicated in [11] is essentially equivalent to the unboundedness of the Hilbert transform in $L^1\left(\mathbb{T}, \frac{d\theta}{2\pi}\right)$.

Finally, most recently, V. Totik [16] has ingeneously sharpened the argument in [11, Thm. 3.4] to show that the asymptotics $O\left(\varepsilon \log \frac{1}{\varepsilon}\right)$, obtained in [11], is in fact sharp in the $A(\mathbb{D})$ context.

However, if one moves away from a "hypodirichlet" algebras context, e.g., replacing in the disk the algebra A $(\mathbb{D})$ on the circle by putting it in the Bergman space context, i.e., $A(\mathbb{D}) \subset C(D) \subset L^1\left(\mathbb{D}, \frac{dA}{\pi}\right)$, the problem has remained untreated. It was mentioned in passing in [10] over two decades ago, but not much progress in understanding the problem in the setting of aggressively non-hypodirichlet case has been achieved. (The set of representing measures for point evaluations in the disk is far from finite-dimensional and is enormous.)

Let us briefly describe the contents of the paper.

In Section 2 we tighten the arguments from [11] and slightly improve the $O\left(\sqrt[4]{\varepsilon}\right)$ estimate to $O\left(\varepsilon^{1/2}\log\frac{1}{\varepsilon}\right)$ for the hypodirichlet case. In view of the Totik result one cannot do any better than $O\left(\varepsilon\log\frac{1}{\varepsilon}\right)$. Yet, it is still an open question whether one can improve the estimates to $O\left(\varepsilon\log\frac{1}{\varepsilon}\right)$ or even $O\left(\sqrt{\varepsilon}\right)$, in the hypodirichlet case. Since the only known examples of hypodirichlet algebras today boil down to analogs of the disk algebra on open finite Riemann surfaces with boundary, we restrict the discussion to finitely connected planar domains to avoid drowning the reader in unnecessary technicalities. Using the techniques from [13] and [11], it is a routine exercise to extend the results to finite Riemann surfaces and, further, to abstract hyperdirichlet algebras.

In Section 3 we try to lay out some results on Wermer's question in the Bergman space context. We'll finish with a discussion of some related open questions in Section 4.

2 ε-L^1-approximation in the hypodirichlet case

Let Ω be a finitely connected domain with the real-analytic boundary $\Gamma := \bigcup\limits_1^n \gamma_j$, $z_0 \in \Omega$, $g(z,z_0)$ is the Green function with the pole z_0. The harmonic measure is defined by $dw = dw_{z_0} = \frac{1}{2\pi}\frac{\partial g}{\partial n_\zeta}(\zeta,z_0)\,ds$, $\zeta \in \Gamma$, $\frac{\partial}{\partial n}$ is the inner normal to Γ at $\zeta \in \Gamma$, ds is the arc length on Γ.

The harmonic measures w_j of the boundary curves $\gamma_1,\dots,\gamma_n$ are defined as solutions of the Dirichlet problem

$$w_j\,|_{\gamma_k} = \delta_{jk}, \quad j,k = 1,\dots,n.$$

(δ_{jk} is the usual Kronecher symbol.) $A := A(\Omega) = \left\{f \in \mathrm{Hol}(\Omega) \cap C\left(\overline{\Omega}\right)\right\}$ is the "disk-algebra".

The Schottky functions $S_j(\zeta) = \frac{\partial w_j(\zeta)}{\partial n_\zeta} \Big/ \frac{\partial g(\zeta,z_0)}{\partial n_\zeta}$, $j = 1,\dots,n$ are real-analytic on Γ and annihilate $\mathrm{Re}\,A$ on Γ. In fact, if u is a harmonic function on Ω, continuous in $\overline{\Omega}$ and v is its harmonic conjugate the period of v around the contour γ_j, $j = 1,\dots,n$, is given by

$$\Delta_{\gamma_j} v = -\int\limits_\Gamma u(\zeta) S_j(\zeta)\,dw$$

$\dim\left\{S_j\right\}_1^n = n-1$. For details we refer the reader to [3,4,8,9,13]. The following is a slight improvement of Thm. 1.2 in [11]. Let $H^p(\Omega)$ denote the Hardy space, $p > 0$, in Ω — cf. [3,4,8], the closure of $A(\Omega)$ in $L^p(dw)$.

Theorem 2 *Let $f \in C(\Gamma)$. Assume that* $\mathrm{dist}_{L^1(dw)}\left(f, H^1\right) \le \varepsilon$*, or, equivalently,* $\exists G \in A$*, s.t.*

$$\int\limits_\Gamma |G - f|\,dw < \varepsilon$$

for some $\varepsilon > 0$*. Then, there exists* $G^* \in A$*,* $\|G^*\|_{C(\Gamma)} \le \|f\|_{C(\Gamma)}$*, such that*

$$\int_\Gamma |G^* - f|\, dw \le C(z_0, \Omega)\, \varepsilon^{1/2} \log \frac{1}{\varepsilon}$$

where the constant C *depends only on* Ω, z_0 *but not on* f*.*

Proof From now on $\|\cdot\|$ stands for $\|\cdot\|_{C(\Gamma)}$, the supremum norm. Without loss of generality $\|f\| = 1$. Let $p : 0 < p < 1$. Following the ideas in [11], define sets

$$\begin{aligned} E_1 &:= \left\{\zeta \in \Gamma; |G(\zeta)| \ge 1 + \varepsilon^{1-p}\right\} \\ E_2 &:= \left\{\zeta \in \Gamma; |G(\zeta)| \le 1 + \frac{\varepsilon^{1-p}}{2}\right\} \\ E_0 &:= \Gamma \setminus E_1 \setminus E_2. \end{aligned}$$

Lemma 1 $\int_{E_1 \cup E_0} |G|\, dw < 3\varepsilon^p$

Proof

$$\begin{aligned} w(E_1 \cup E_0) &= w\left(\left\{\zeta \in \Gamma : |G(\zeta)| \ge 1 + \frac{\varepsilon^{1-p}}{2}\right\}\right) \\ &\le \int_{E_1 \cup E_0} (|G| - 1) \cdot \frac{2}{\varepsilon^{1-p}}\, dw \le \frac{2}{\varepsilon^{1-p}} \int_{E_1 \cup E_0} (|G| - 1)\, dw \\ &\le \frac{2}{\varepsilon^{1-p}} \int_{E_1 \cup E_0} |G - f|\, dw \le \frac{2}{\varepsilon^{1-p}}\, \varepsilon = 2\varepsilon^p \end{aligned}$$

Hence,

$$\begin{aligned} \int_{E_1 \cup E_0} |G|\, dw &\le \int_{E_1 \cup E_0} (|G| - |f| + 1)\, dw \\ &\le \int_\Gamma |G - f|\, dw + w(E_1 \cup E_0) \le \varepsilon + 2\varepsilon^p \le 3\varepsilon^p. \end{aligned}$$

Lemma 2 *For* $\delta > 0$ *sufficiently small define* $u_0 \in C_{\mathbb{R}}(\Gamma)$ *as follows:*

$$\begin{aligned} u_0 &\ge \delta \quad \text{on } \Gamma \\ u_0\, |_{E_2} &\equiv \delta, \\ u_0\, |_{E_1} &\equiv \log |G|, \\ u_0\, |_{\Gamma \setminus E_2} &\le \log |G| + \frac{\delta}{2}. \end{aligned}$$

(*This is, of course, possible by Urysohn's lemma.*) *Then, there exists* $u_1 \in C(\Gamma)$ *such that* $u_1 \ge 0$*,* $\|u_1\| \le C(z_0, \Omega)\,(\varepsilon^p + \delta)$ *and* $u_0 + u_1 \in \left(\overline{\operatorname{Re} A}\right)_{C(\Gamma)}$ *= the uniform closure of the real parts of A.*

Since the proof is essentially identical to that of Lemma 2.2 in [11], we only indicate the main steps (also cf. to Lemma 2 in [9]).

Define the linear map $\Lambda : C_{\mathbb{R}}(\Gamma) \to \mathbb{R}^{n-1}$,

$$\Lambda : u \mapsto \left(\int_{\Gamma} S_j u\, dw \right)_{j=1}^{n-1}$$

(Here and on we assume γ_n is the outer boundary complement.) Let $K_+ = \{u \in C_{\mathbb{R}}(\Gamma) : u \geq 0\}$ be the positive cone in $C_{\mathbb{R}}(\Gamma)$.

Claim (Claim 1) Λ is a bounded linear operator and $\Lambda\,(K_+) = \mathbb{R}^{n-1}$.

We omit the proof of Claim 1 since it is identical to that in [9,11].

Claim (Claim 2) Set $\|x\|_{\mathbb{R}^{n-1}} = \max\limits_{1<j<n-1} \left|x_j\right|, x = (x_1 \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Then, $\|\Lambda\,(u_0)\|_{\mathbb{R}^{n-1}} \leq C_{z_0}\,(\varepsilon^p + \delta)$.

For each $j = 1, \ldots, n-1$

$$\left| \int_{\Gamma} u_0 S_i\, dw \right| \leq \max_{1 \leq j \leq n-1} \left\|S_j\right\| \int_{\Gamma} u_0\, dw = C_{z_0} \left\{ \int_{E_2} u_0 + \int_{E_1 \cup E_0} u_0 \right\}$$

and the claim follows from Lemma 1 and elementary inequalities $\log^+ x \leq x$, $\varepsilon^p \geq \varepsilon$, $0 < p < 1$, $\varepsilon < 1$.

To finish the proof of Lemma 2, find by Claim 1, $u_1 \in K_+ : \Lambda\,(u_1) = -\Lambda\,(u_0)$.

Since $\Lambda\,(K_+) = \mathbb{R}^{n-1}$, $\exists C_{z_0} : \Lambda\left(\left\{u \in K_+ : \|u\| \leq C_{z_0}\right\}\right) \supset \left\{x \in \mathbb{R}^{n-1} \|x\|_{\mathbb{R}^{n-1}} \leq C'_{z_0}\right\}$. Then,

$$\begin{aligned} &\Lambda\left(\left\{u \in K_+ : \|u\| \leq C'_{z_0}\,(\varepsilon^p + \delta)\right\}\right) \\ &\supset \left\{x \in \mathbb{R}^{n-1} : \|x\|_{\mathbb{R}^{n-1}} \leq C_{z_0}\,(\varepsilon^p + \delta)\right\}, \end{aligned}$$

where C_{z0} is the same as in Claim 2.

So, $\exists u_1 \geq 0$, $\Lambda\,(u_1) = -\Lambda\,(u_0)$ and $\|u_1\| \leq C'_{z_0}\,(\varepsilon^p + \delta)$.

Consider the function $u_0 + u_1$, $\Lambda\,(u_0 + u_1) = 0$, so it is orthogonal to the Span $\left\{S_j\right\}_{j=1}^{n-1}$. Then, obviously, $u_0 + u_1 \in \operatorname{Re} H^2(\Omega) \cap C_{\mathbb{R}}(\Gamma)$, hence $u_0 + u_1 \in \overline{\operatorname{Re} A}$. Lemma 2 is proved.

Now, continue the proof of Thm. 2. Fix $\delta < \frac{\varepsilon^p}{2}$. Let u_0, u_1 be the same as in Lemma 2 and take $u \in \operatorname{Re} A$ such that $\|u - u_0 - u_1\| \leq \frac{\delta}{2}$. Then, $u \geq \frac{\delta}{2} > 0$ on Γ since $u_0 + u_1 \geq u_0 \geq \delta$ on Γ.

Set $G_0 = Ge^{-(u+iv)}$, where $u + iv \in A$, $v\,(z_0) = 0$. Clearly, $G_0 \in A$.

Claim (Claim 3) $|G_0| \leq 1 + \varepsilon^{1-p}$ on $E_2 \cup E_0$, and $|G_0| \leq 1 + \varepsilon^p$ on E_1.

Indeed,

$$|G_0|_{E_2\cup E_0} \le |G|_{E_2\cup E_0} \le 1+\varepsilon^{1-p}.$$

On E_1, we have (cf. the definition of E_1):

$$\begin{aligned}|G_0| &= |G|e^{-u} = |G|e^{-u_0-u_1}e^{(u_0+u_1-u)}\\ &\le |G|e^{-\log|G|}e^{\|u_0+u_1-u\|} \le 1+2\delta \le 1+\varepsilon^p\end{aligned}$$

assuming that ε and, hence, δ are small enough.

Now,

$$\int\limits_{\Gamma} |G_0 - f|\,dw \le \int\limits_{\Gamma} |G_0 - G|\,dw + \int\limits_{\Gamma} |G-f|\,dw \le \int\limits_{\Gamma} |G_0 - G| + \varepsilon. \tag{1}$$

Also,

$$\begin{aligned}\int\limits_{\Gamma} |G_0 - G|\,dw &\le \int\limits_{\Gamma} |G_0|\left|\frac{G}{G_0} - 1\right| dw\\ &\le \left(1+\varepsilon^{1-p}\right)\int\limits_{E_2\cup E_0} \left|e^{u+iv}-1\right| dw + (1+\varepsilon^p)\int\limits_{E_1} \left|e^{u+iv}-1\right| dw.\end{aligned} \tag{2}$$

Claim (Claim 4) $\int_\Gamma \left|e^{u+iv}\right| dw \le 1 + C_{z_0}\varepsilon^p$.

Proof (Proof of Claim 4) Using Lemmas 1 and 2 and the definition of u_0, we obtain for ε, δ sufficiently small:

$$\begin{aligned}\int\limits_{\Gamma} \left|e^{u+iv}\right| dw &= \int\limits_{E_1\cup E_0} e^u dw + \int\limits_{E_2} e^u dw\\ &\le \int\limits_{E_1\cup E_0} e^{u_0 + C_{z_0}(\varepsilon^p+\delta)+\frac{\delta}{2}}dw + \int\limits_{E_2} e^{u_0 + C_{z_0}(\varepsilon^p+\delta)+\frac{\delta}{2}}dw\\ &\le \int\limits_{E_1\cup E_0} e^{\log|G| + C_{z_0}(\varepsilon^p+\delta)+\delta}dw + \int\limits_{E_2} e^{\frac{3}{2}\delta + C_{z_0}(\varepsilon^p+\delta)}dw\\ &\le C'_{z_0}\int\limits_{E_1\cup E_0} |G|\,dw + \int\limits_{E_2}\left(1+C''_{z_0}\varepsilon^p\right)dw \le 1 + C'''_{z_0}\varepsilon^p,\end{aligned}$$

where $C_{z_0}, C'_{z_0}, C''_{z_0}$, depend only on z_0.

By Claim 4,

$$\int\limits_{\Gamma} \left|e^{u+iv}\right| dw = \int\limits_{\Gamma} e^u dw \le 1 + C_{z_0}\varepsilon^p, \tag{3}$$

so, $\left(u > \frac{\delta}{2} > 0\right)$,

$$\int\limits_{\Gamma} (e^u - 1)\,dw = \int\limits_{\Gamma} \left|e^{u+iv} - e^{iv}\right| dw \le C_{z_0}\varepsilon^p. \tag{4}$$

Let's estimate

$$\int_{\Gamma} \left|e^{iv}-1\right| dw = \int_{\Gamma} \left|2\sin\frac{v}{2}\right| dw.$$

From (3), Jensen's inequality and an elementary inequality $\log(1+x) \le x$ for $x \ge 0$, we obtain

$$\int_{\Gamma} u\, dw \le C_{z_0}\varepsilon^p. \tag{5}$$

Now, by general Kolmogorov's inequality,

$$\begin{aligned}\int_{\{\zeta:|v(\zeta)|\ge 1\}} \left|2\sin\frac{v}{2}\right| dw &\le 2w\{\zeta : |v(\zeta)| \ge 1\}\\ &\le C\|u\|_{L^1(dw)} \le C\varepsilon^p.\end{aligned}$$

Further, for any $q : 0 < q < 1$ and $|v| \le 1$, we have $2\left|\sin\frac{v}{2}\right| \le |v| \le |v|^q$ and using general V. I. Smirnov's theorem ([4, Ch. 3, Thm. 2.4]), since $u > 0$, we obtain from (5):

$$\begin{aligned}\int_{\{\zeta:|v(\zeta)|\le 1\}} \left|2\sin\frac{v}{2}\right| dw &\le \frac{1}{\cos\left(q\frac{\pi}{2}\right)}\|u\|^q_{L^1(dw)}\\ &\le \frac{C_{z_0}\varepsilon^{pq}}{1-q}.\end{aligned}$$

Thus, since $\left|e^{u+iv}-1\right| \le \left|e^{u+iv}-e^{iv}\right| + \left|e^{iv}-1\right|$, we obtain

$$\begin{aligned}\int_{\Gamma} \left|e^{u+iv}-1\right| dw &\le \int_{\Gamma} (e^u-1)\, dw + \int_{\Gamma} \left|e^{iv}-1\right| dw\\ &\le C'_{z_0}\varepsilon^p + C_{z_0}\frac{\varepsilon^{pq}}{1-q},\end{aligned} \tag{6}$$

where q, p were arbitrary positive numbers < 1.

Now from (1), (2) and (6) we infer that there exists an absolute constant $C = C(z_0, \Omega)$ such that

$$\int_{\Gamma} |G_0 - f|\, dw \le C\left(\varepsilon^p + \frac{\varepsilon^{pq}}{1-q}\right), \tag{7}$$

where $0 < p, q < 1$ and $\|G_0\| \le \max\left(1+\varepsilon^{1-p}, 1+\varepsilon^p\right) \le 1+\varepsilon^p$ for $p : 0 < p \le \frac{1}{2}$. The minimum of $\frac{(\varepsilon^p)^q}{1-q}$ over all $0 < q < 1$ is $e\varepsilon^p\log\frac{1}{\varepsilon^p}$. Thus $\|G_o - f\|_{L^1(dw)} \le C\left(\varepsilon^p + \varepsilon^p\log\frac{1}{\varepsilon^p}\right)$, where C is a constant. The function $\varphi(p) := \varepsilon^p\left(1+p\log\frac{1}{\varepsilon}\right)$ decreases for $p : 0 < p \le \frac{1}{2}$ and attains its minimum $\varepsilon^{1/2}\left(1+\frac{1}{2}\log\frac{1}{\varepsilon}\right)$ at $p = 1/2$. Hence, $\|G_o - f\|_{L^1(dw)} \le C\left(\varepsilon^{1/2} + \varepsilon^{1/2}\log\frac{1}{\varepsilon}\right) \le C'\varepsilon^{1/2}\log\frac{1}{\varepsilon}$. Finally, replacing G_o by $G^* := G_o/1+\varepsilon^{1/2}$, $\|G^*\| \le 1$, we finish the proof of the theorem.

Remark 1 As the reader has undoubtedly noted, we could have fixed $p = 1/2$ at the very beginning and arrived at the same statement at the end.

The reason we painstakingly carried out the argument with an arbitrary p was the hope that at the final stage the variation with respect to p will provide a sharper estimate $O\left(\varepsilon \log \frac{1}{\varepsilon}\right)$. At this point we don't see how to do it, nor were we able to get rid of the logarithm and obtain at least $O\left(\sqrt{\varepsilon}\right)$ estimate. We've left the general scheme in tact in the hope that one reader might be more successful along these lines.

It is tempting to conjecture that the correct asymptotics might be $O\left(\varepsilon \log \frac{1}{\varepsilon}\right)$ even in the hypodirichlet case. However, there is a difference between this case and the Dirichlet algebra case treated in Thm. 3.3 in [11]. The problem is that Lemma 2 is immediate in the Dirichlet algebra case and the "buffer" set E_o in Lemma 1 is not needed — cf. the proof of Thm. 3.3 in [11].

It would be a worthy project to try to follow the steps in the proof more or less explicitly, when Ω is an annulus. We haven't been able to obtain a better estimate.

3 Wermer's Question in the Bergman Norm

Unfortunately we haven't been able to settle Question 1.2 in a satisfactory fashion in the context of the Bergman space $A^1 \subset L^1\left(\mathbb{D}, \frac{dA}{\pi}\right)$ with respect to $B^1 = \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_\infty \le 1\}$. Here are some results we have been able to establish.

Proposition 1 *Let $f \in C\left(\overline{\mathbb{D}}\right)$. Assume without loss of generality that $\|f\| = 1$, i.e., the uniform norm of f is 1. There exists a function $\varphi : [0,1] \to \mathbb{R}_+$, $\varphi(0) = 0$ continuous at the origin, such that if $\mathrm{dist}_{L^1(\mathbb{D})}\left(f, A^1\right) \le \varepsilon$, then $\mathrm{dist}_{L^1(\mathbb{D})}\left(f, B^1\right) \le \varphi(\varepsilon)$.*

Proof Suppose to the contrary, $\exists\, \delta > 0$ such that we can find $\{w_n\} \in L^\infty$, $\|w_n\|_\infty = 1$, $\varepsilon_n \downarrow 0$ such that $\mathrm{dist}_{L^1(\mathbb{D})}\left(w_n, A^1\right) \le \varepsilon_n$ but $\mathrm{dist}_{L^1(\mathbb{D})}\left(w_n, B^1\right) \ge \delta$. Without loss of generality, taking subsequences, we can assume that by the Banach–Alaoglu theorem $w_n \to w$ in the weak $(*)$ topology of L^∞. First show that $w \in B^1$. Indeed, $\|w\|_\infty \le \liminf \|w_n\|_\infty \le 1$. Thus, we only need to show that w coincides a.e. with an analytic function. Take any $\varphi \in C_0^\infty(\mathbb{D})$. Let $f_n \in A^1 : \mathrm{dist}_{L^1(\mathbb{D})}(w_n, f_n) \le \varepsilon_n$. We have, using integration by parts,

$$\int_{\mathbb{D}} w \cdot \frac{\partial \varphi}{\partial \bar{z}}\, dA = \lim_{n\to\infty} \int_{\mathbb{D}} w_n \frac{\partial \varphi}{\partial \bar{z}}\, dA = \lim_{n\to\infty} \int_{\mathbb{D}} (w_n - f_n) \frac{\partial \varphi}{\partial \bar{z}}\, dA.$$

So,

$$\left| \int_{\mathbb{D}} w \frac{\partial \varphi}{\partial \bar{z}}\, dA \right| \le \lim_{n\to\infty} \left\| \frac{\partial \varphi}{\partial \bar{z}} \right\| \cdot \|w_n - f_n\|_{L^1} = 0.$$

Thus, from Weyl's lemma, cf., e.g., §1 in [15] and references therein, it follows that $w \in H^\infty$, $\|w\|_\infty \le 1$, so $w \in B^1$.

Yet, by our assumption $\|w_n - w\|_{L^1(\mathbb{D})} \ge \delta$. Since $\|w_n - f_n\|_{L^1(\mathbb{D})} \to 0$, $\|f_n - w\|_{L^1(\mathbb{D})} \ge \frac{\delta}{2}$. Once again, going to subsequences, we can assume that $f_n \to f \in A^1$ weakly in

$L^1(\mathbb{D})$ (cf. Thm. 8.2 and Cor. 2 in [15]). We can do it since obviously, $\|f_n\|_{A^1} \le M < +\infty$. At the same time,

$$0 = \mathrm{weak}(*)_{L^\infty} \lim (w_n - w) = \mathrm{weak}_{L^1} \lim_{n\to\infty} (w_n - f_n + f_n - w) = f - w.$$

So, $f = w$. To summarize, $f_n \to w$ weakly in L^1, so, (cf. §8.3 in [15]) $f_n \to w$ uniformly on compact subsets of $\mathbb{D}$. Moreover, $\{f_n\}^\infty$ are uniformly integrable because for any measurable set $E \subset \mathbb{D}$,

$$\int_E |f_n|\, dA \le \int_E |f_n - w_n|\, da + \int_E |w_n|\, dA \le \varepsilon_n + \mathrm{Area}(E).$$

Thus, by Vitali's theorem, $f_n \to w$ in L^1. This contradicts to $\|f_n - w\|_{L^1} \ge \delta/2$. This ends the proof of the proposition.

Recall from [7] that if X is a metric space, $Y \subset X$ is a subset, the metric projection of $x \in X$ on Y is the set

$$\mathscr{P}(x) = \{y \in Y : \mathrm{dist}_X(x,y) = \mathrm{dist}_X(x,Y)\}.$$

Of most interest, [7], is the situation when Y is closed and $\#\{\mathscr{P}(x)\} = 1$.

In [7], it is shown that for $X = L^1(\mathbb{T})$, $Y = H^1(\mathbb{D})\,|_{\mathbb{T}} \subset L^1(\mathbb{T})$, such metric projection $L^1(\mathbb{T}) \ni f \mapsto g \in H^1$, g is the best approximation to f in H^1, is well-defined, i.e., the best approximation is unique (which was known before, from the results of S. Ya. Khavinson, W. W. Rogosinski and H. S. Shapiro — cf. the references in [10]) and continuous.

The situation for the pair $L^1(\mathbb{D}) \supset A^1$ is more complex.

First of all for non-continuous functions in $L^1(\mathbb{D})$ the metric projection map are not well-defined.

The following example is from [10, Ex. 3.2].

Let $w = \chi_{D_0}$, the characteristic function of $D_0 = \left\{z \in \mathbb{D} : |z| < 1/\sqrt{2}\right\}$. Then, $f^* = c$ for any $c : 0 \le c \le 1$ gives the best A^1-approximation to w. The reason is that if we take $g^* = \begin{cases} 1, & z \in D_0 \\ -1, & z \in \mathbb{D} \setminus D_0 \end{cases}$ then

$$\int_{\mathbb{D}} g^* z^n\, dA = 0, \quad n = 0,1,2,\ldots.$$

So, g^* annihilates A^1, $g^* \in L^\infty$ and for any $c : 0 \le c \le 1$ we have

$$g^*(w-c) = |w-c| \text{ a.e. in } \mathbb{D}.$$

This, of course, is Hahn–Banach characterization of the best approximation — cf. [10] for details. However, as is shown in [10], the best A^1-approximation to $w \in L^1(\mathbb{D})$ is unique provided, e.g., that $w \in C(\mathbb{D} \setminus E)$ and E is relatively closed in $\mathbb{D}$ and does not separate $\mathbb{D}$ (cf. [10], also see the discussion of even stronger results there going back to S. Ya. Khavinson). When the metric projection is well-defined for $w \in L^1(\mathbb{D}, dA)$, it is in fact continuous. The latter, following the generic reasons in [7], is a corollary of the following fact. (Note that in $L^1 - H^1$ context this was proved by D. J. Newman in [14].)

Proposition 2 *Let $w \in L^1(dA)$, $\{f_n\} \in A^1$, and $\|w - f_n\|_{L^1(\mathbb{D})} \to d := \mathrm{dist}_{L^1}\left(w, A^1\right) =: \|w - f^*\|_{L^1}$, i.e., f^* is the best A^1-approximation to w. Assume that f^* is the unique best approximation to w in A^1.*

Then, $\|f_n - f^*\|_{L^1(\mathbb{D})} \to 0$ as $n \to \infty$.

Proof Suffices to show that any subsequence of $\{f_n\}_1^\infty$ contains another subsequence converging in L^1 to f^*. Obviously, $\|f_n\|_{L^1(\mathbb{D})} \le M < +\infty$.

Taking any subsequence of $\{f_n\}_1^\infty$ that we, hopefully without confusion, will still denote $\{f_n\}$.

Then, we can extract a subsequence $\left\{f_{n_k}\right\}_{k=1}^\infty$ such that the measures $f_{n_k} dA \xrightarrow{\text{weak}^*} F\,dA$. By the F. and M. Riesz theorem for Bergman spaces [15, Thm. 8.2], $F \in A^1$. Clearly, invoking the Bergman reproducing kernel $K(z,\xi) = \frac{1}{\pi}\frac{1}{(1-z\xi)^2}$, $f_{n_k}(z) \to F(z)$, $z \in \mathbb{D}$, and the convergence is uniform on compact subsets of $\mathbb{D}$. Thus, Fatou's lemma yields

$$d \le \|w - F\|_{L^1} \le \liminf_{k\to\infty} \left\|w - f_{n_k}\right\|_{L^1} = d.$$

So, $F = f^*$, since w has unique best A^1-approximation. Hence, we have

$$\left\|w - f_{n_k}\right\|_{L^1(\mathbb{D})} \to \|w - f^*\|_{L^1(\mathbb{D})} \text{ and } w - f_{n_k} \to w - f^*$$

pointwise. By the well-known theorem of real analysis (G. M. Fikhtengoltz' theorem as it is called in the Russian literature), $w - f_{n_k} \xrightarrow{L^1(\mathbb{D})} w - f^*$, so $f_{n_k} \xrightarrow{L^1(\mathbb{D})} f^*$, as we set to prove. □

Remark 2 Unfortunately, there are no known sufficient criteria for everywhere discontinuous bounded functions for which the metric projection onto A^1 would still be well defined. In the opposite direction, there are no known examples of non-radial functions w with massive sets of discontinuities for which the best A^1-approximation is unique.

The following rough result provides the negative answer to Wermer's question if one limits oneself to the $O(\varepsilon)$ asymptotics. Recall that $B^1 := \{f \in H^\infty : \|f\|_\infty \le 1\}$.

Theorem 3 *For all $\delta > 0$ sufficiently small there exist functions $\eta_\delta \in C\left(\overline{\mathbb{D}}\right)$, $\|\eta_\delta\|_\infty = 1$, such that $\mathrm{dist}_{L^1(\mathbb{D})}\left(\eta_\delta, A^1\right) \le \delta$, but $\frac{1}{\delta}\,\mathrm{dist}_{L^1(\mathbb{D})}\left(\eta_\delta, B^1\right) \to \infty$ when $\delta \to 0$.*

Proof Let $\Omega := \{x + iy : x > 0, y < q(x)\}$, where $q(x)$ is a smooth, convex, decreasing function, $q(0) = m \ll 1$, $\lim\limits_{x\to\infty} q(x) = 0$. We shall leave out for now more precise specifications of q and the rate of decrease of $q(x)$.

Let $f := u + iv$ be the Riemann map of $\mathbb{D}$ onto Ω, once again, leaving for later the normalization $f(0)$, let $f(1) = (+\infty, 0)$.

Fix $\varepsilon > 0$. Let $\varphi_\varepsilon : [0,1) \mapsto [0,2]$ be a continuous function satisfying the following properties

(i) $\varphi_\varepsilon(r) = 0$, $1 - \varepsilon \le r < 1$;

(ii) $\int\limits_0^1 \mathrm{sgn}\left(\varphi_\varepsilon(r) - 1\right) r\,dr = 0$, (sgn, as always, denotes the signum function);

(iii) $\int_0^1 |\varphi_\varepsilon(r)-1|\,r\,dr \le C\varepsilon$, where C is sufficiently large and fixed.

(It's a trivial exercise to check that $C=2$ suffices.)

Finally, set $w_\varepsilon(z)=\varphi_\varepsilon(|z|)u(z)+iv(z)$. Note that $w_\varepsilon(z)-f(z)=(\varphi_\varepsilon(|z|)-1)\,u(z)$. Thus, invoking property (ii), the fact that $\varphi_\varepsilon(|z|)-1$ is a radial function, while by the mean value property

$$\int_0^{2\pi} u\left(re^{i\theta}\right)d\theta = 2\pi\, u(0)$$

and $u>0$, we conclude via integration in polar coordinates that

$$\int_{\mathbb{D}} \operatorname{sgn}\left(\varphi(|z|)-1\right)u(z)\,z^n\,dA=0, \quad n=0,1,2,\dots.$$

Hence, as before (cf. [10]), the Hahn–Banach characterization of the best approximation in L^1 yields that $f(z)$ is in fact the best A^1-approximation to $w_\varepsilon(z)$ in L^1.

Moreover, from (iii), using again the mean value property one easily derives that

$$\operatorname{dist}_{L^1(\mathbb{D},dA)}\left(w_\varepsilon,A^1\right)=\|w_\varepsilon-f\|_{L^1(\mathbb{D})}=u(0)\,O(\varepsilon).$$

Now, since $v(z)$ is continuous in $\overline{\mathbb{D}}$ and $u(z)$ is its harmonic conjugate, $u(z)\sim o(1)\log|1-z|$, where $o(1)$ depends only on m and the rate of decent of $q(x)$, the function defining the boundary of Ω. Therefore $\|w_\varepsilon\|_{L^\infty}=o\left(\log\frac{1}{\varepsilon}\right)$ (cf. property (i)). Now, if we rescale and set $\eta_\varepsilon := \frac{w_\varepsilon}{\|w_\varepsilon\|_\infty}$, $f_\varepsilon=\frac{f}{\|w_\varepsilon\|_\infty}$, we have for a fixed $\varepsilon>0$:

$$\eta_\varepsilon\in C\left(\overline{\mathbb{D}}\right), \quad \|\eta_\varepsilon\|_\infty=1, \quad \operatorname{dist}_{L^1(\mathbb{D})}\left(\eta_\varepsilon,A^1\right)\sim\frac{u(0)\,\varepsilon}{\|\eta_\varepsilon\|_{L^\infty}}.$$

Observe that $\exists c_1,c_2$ independent positive constants, such that (w_ε is supported on $\{|z|\le 1-\varepsilon\}$

$$c_1\|w_\varepsilon\|_\infty \le \max_{|z|\le 1-\varepsilon} u \le c_2\|w_\varepsilon\|_\infty.$$

and as pointed out earlier, $\max_{\{|z|\le 1-\varepsilon\}} u\sim o\left(\log\frac{1}{\varepsilon}\right)=\alpha(\varepsilon)\cdot\log\frac{1}{\varepsilon}$, $\alpha(\varepsilon)\to 0$ with ε and $\frac{1}{\alpha(\varepsilon)\log\frac{1}{\varepsilon}}\to 0$.

Let us narrow down the choice of Ω. By Chebyshev's inequality,

$$\frac{\text{Area}\{z\in\mathbb{D}:u(z)\ge\lambda>0\}}{\pi}\le\int_{\{u(z)\ge\lambda\}}\frac{u(z)}{\lambda}\frac{dA}{\pi}\le\frac{u(0)}{\lambda}$$

Now, we choose the "upper boundary" $\{y=q(x)\}$ of Ω so that for some constant $c=c(q)$:

$$\frac{1}{\pi}\text{Area}\{z\in\mathbb{D}:u(z)\ge\lambda>0\}\ge\frac{c}{\lambda}\,\beta(\lambda),$$

where $\beta(\lambda)\to 0$, when $\lambda\to\infty$. ($\beta(\lambda)$ can be chosen in fact arbitrarily, of course, $c=c(q)$ will depend on $\beta(\lambda)$.) Basically, it means that $q(x)\to 0$ at ∞ arbitrarily slow. Finally, we normalize $f(0)=1+i\,\frac{m}{2}$.

We have now

$$\operatorname{dist}_{L^1(\mathbb{D})}\left(\eta_\varepsilon, A^1\right) \simeq \frac{u(0)}{\|w_\varepsilon\|_\infty}\,\varepsilon = \frac{\varepsilon}{\|w_\varepsilon\|_\infty} = \|\eta_\varepsilon - f_\varepsilon\|_{L^1(\mathbb{D})}$$

Take an arbitrary $\Psi \in B^1$, so $|\Psi| \leq 1$. We have, since $|\Psi| \leq 1$:

$$\begin{aligned}
&\|\eta_\varepsilon - \Psi\|_{L^1(\mathbb{D})} \geq \|f_\varepsilon - \Psi\|_{L^1(\mathbb{D})} - \|f_\varepsilon - \eta_\varepsilon\|_{L^1(\mathbb{D})} \\
&= \|f_\varepsilon - \Psi\|_{L^1(\mathbb{D})} - \frac{\varepsilon}{\|w_\varepsilon\|_\infty} \geq \int_{\{z:|f_\varepsilon|\leq 1\}} |f_\varepsilon - \Psi|\, dA + \int_{\{z:|f_\varepsilon|\geq 2\}} |f_\varepsilon - \Psi|\, dA \\
&\quad - \frac{\varepsilon}{\|w_\varepsilon\|_\infty} \geq \operatorname{Area}\{z : |f_\varepsilon| \geq 2\} - \frac{\varepsilon}{\|w_\varepsilon\|_\infty} \\
&\quad \geq \operatorname{Area}\left\{z : u(z) \geq 2\alpha(\varepsilon)\log\frac{1}{\varepsilon}\right\} - \frac{\varepsilon}{\alpha(\varepsilon)\cdot\log\frac{1}{\varepsilon}} \\
&\quad \geq \frac{c\beta\left(\alpha(\varepsilon)\log\frac{1}{\varepsilon}\right)}{\alpha(\varepsilon)\log\frac{1}{\varepsilon}} - \frac{\varepsilon}{\alpha(\varepsilon)\log\frac{1}{\varepsilon}}.
\end{aligned}$$

The proof is complete after we choose β so that $\frac{\beta\left(\alpha(\varepsilon)\log\frac{1}{\varepsilon}\right)}{\varepsilon} \to \infty$ when $\varepsilon \to 0$ and set $\delta := \frac{\varepsilon}{\alpha(\varepsilon)\log\frac{1}{\varepsilon}} = \operatorname{dist}_{L^1(\mathbb{D})}\left(\eta_\varepsilon, A^1\right)$.

4 Final Remarks

(I) As is clear from the last part of the proof of Thm. 3, the function β responsible for the rate of convergence of $\frac{\operatorname{dist}\left(\eta_\delta, B^1\right)}{\operatorname{dist}\left(\eta_\delta, A^1\right)} \to \infty$ when $\delta \downarrow 0$ is, more or less, arbitrary. Thus, in essence, the answer to Wermer's question is in the negative in $L^1(\mathbb{D}) - A^1(\mathbb{D})$ setting. Moreover though, according to Prop. 1, for any $f : \|f\|_\infty \leq \operatorname{dist}_{L^1(\mathbb{D})}\left(f, A^1\right) \leq \varepsilon \Rightarrow \operatorname{dist}_{L^1(\mathbb{D})}\left(f, B^1\right) \leq \varphi(\varepsilon)$ with $\varphi(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$, the rate at which $\varphi(\varepsilon) \downarrow 0$ can be arbitrary slow compared to $\varepsilon \downarrow 0$. A very different situation from $L^1 - H^1$ on $\mathbb{T} = \partial\mathbb{D}$, where (cf. [11, Thm. 3.2]) $\varphi(\varepsilon)$ can be chosen to be $O\left(\varepsilon\log\frac{1}{\varepsilon}\right)$, "almost as good" as $O(\varepsilon)$.

(II) If we replace $f(z)$ in the proof of Thm. 3 by an (infinite) convex combination of rotates of f by all rational angles, fix ε, say $\frac{1}{2}$, and consider w built as before based on f, we shall get a (known) example of a function in $C\left(\overline{\mathbb{D}}\right)$ whose best A^1 approximation (in $L^1(\mathbb{D}, dA)$) is unbounded near every point of $\mathbb{T} = \partial\mathbb{D}$ (cf. [10, Prop. 7.6]).

A similar example can be built in $L^1 - H^1$ situation on $\mathbb{T}$. Indeed, following [7, p. 99] consider a Taylor series $f_0\left(re^{i\theta}\right) = u_0\left(re^{i\theta}\right) + iv_0\left(re^{i\theta}\right)$ with $u \geq 0$, unbounded near $r = 1$, $\theta = 0$, and $v_0 \in C\left(\overline{\mathbb{D}}\right); v_0(0) = 0$.

As above, translating f by all rational angles θ_n and taking a convex combination of all $f(\theta - \theta_n)$ with rapidly decreasing coefficients $C_n \geq 0$, $\sum_0^\infty C_n = 1$ we obtain the function

$$F(z)=\sum_{0}^{\infty}C_n\left(u_n\left(re^{i\theta}\right)+iv_n\left(re^{i\theta}\right)\right),\ u_n+iv_n=f_0\left(re^{(\theta-\theta_n}\right),$$

clearly in H^1, Re $F\geq 0$ and unbounded near every point on $\mathbb{T}$. $F(z)=F_1(z)+iF_2(z)$, $F_2(0)=0$ Let $\varphi^*:=(-F-F(0))\in H_0^1:=\left\{g\in H^1(\mathbb{D}):g(0)=0\right\}$. $w:=F_1+\varphi^*=F_1-F_1-iF_2+F(0)\in C(\mathbb{T})$, $w-\varphi^*=F_1\geq 0$, so

$$(w-\varphi^*)=|w-\varphi^*|\ \text{ on }\mathbb{T}\text{ a.e.}$$

Therefore, again by the Hahn–Banach duality criterion (cf. [10], e.g.) φ^* is the best H_0^1 approximation in $L^1(\mathbb{T},d\theta)$ to a continuous function $w\in C(\mathbb{T})$ that is unbounded near every point on $\mathbb{T}$. Replacing w by $e^{-i\theta}w=:w_1$ and φ^* by $\frac{\varphi^*}{z}=i\Phi^*$, we get a continuous function on $\mathbb{T}$ whose best L^1-approximation in H^1 is unbounded at every point on $\mathbb{T}$. Obviously, in L^1-H^1-case, this cannot occur if w is smoother than a merely $C(\mathbb{T})$ function, e.g., if $w\in\operatorname{Lip}\alpha(\mathbb{T})$, $\alpha>0$, cf. [7]. In the $L^1(\mathbb{D})-A^1(\mathbb{D})$ situation it is already not known, whether, say $w\in\operatorname{Lip}1$ on $\overline{\mathbb{D}}$ can produce an unbounded, or even discontinuous in $\overline{\mathbb{D}}$ best A^1-approximation in $L^1(\mathbb{D})$ — cf. [10, Question 2].

(III) Along the same lines the question of hereditory regularity of the best (in $L^1(\mathbb{D})$) A^1-approximation to $w\in C^\infty\left(\overline{\mathbb{D}}\right)$ remains unanswered. For example, if $w(z)$ is real-analytic in $\overline{\mathbb{D}}$, does it imply that its best approximant in A^1 is merely continuous in $\overline{\mathbb{D}}$? In the case of L^1-harmonic approximation, it is known to be false [10]. The only result that is known for analytic approximation is that the best A^1-approximation to a $C^\infty\left(\overline{\mathbb{D}}\right)$ function is in $\bigcap\limits_{p>0}H^p$ [10, Thm. 4.1]. This is far from satisfactory.

In L^1-H^1-approximation, it has been known for over 60 years that if w is real-analytic on an arc $\gamma\subset\mathbb{T}$, its best $H^1(\mathbb{D})$-approximation is also real analytic across γ — cf. [12] and references therein.

(IV) As one sees while comparing proofs of Thms. 2 and 3, the fact how large the set of representing measures is for the disk algebra plays a crucial role. More or less equivalently, the size of the annihilator figures out decisively. The latter, of course, is enormous when we consider the disk algebra inside the algebra of continuous functions on the disk rather than the circle. In view of this it would be of interest to consider the above questions in higher dimensions. To the best of the author's knowledge nothing has been done in that setting.

Acknowledgements The author gratefully acknowledges the support of the Simons Foundation.

References

1. Duren, P., Schuster, A.: Bergman Spaces, *Mathematical Surveys and Monographs*, vol. 100. American Mathematical Society, Providence, RI (2004). URL https://doi.org/10.1090/surv/100
2. Duren, P.L.: Theory of H^p Spaces, *Pure and Applied Mathematics*, vol. 38. Academic Press, New York-London (1970)
3. Fisher, S.D.: Function Theory on Planar Domains. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York (1983). A second course in complex analysis, A Wiley-Interscience Publication

4. Gamelin, T.W.: Uniform Algebras. Prentice-Hall, Inc., Englewood Cliffs, N. J. (1969)
5. Garnett, J.B.: Bounded Analytic Functions, *Pure and Applied Mathematics*, vol. 96. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London (1981)
6. Hoffman, K.: Banach Spaces of Analytic Functions. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N. J. (1962)
7. Kahane, J.P.: Best approximation in $L^1(T)$. Bull. Amer. Math. Soc. **80**, 788–804 (1974). URL https://doi.org/10.1090/S0002-9904-1974-13518-4
8. Khavinson, D.: Factorization theorems for different classes of analytic functions in multiply connected domains. Pacific J. Math. **108**(2), 295–318 (1983). URL http://projecteuclid.org/euclid.pjm/1102720365
9. Khavinson, D.: On removal of periods of conjugate functions in multiply connected domains. Michigan Math. J. **31**(3), 371–379 (1984). URL https://doi.org/10.1307/mmj/1029003081
10. Khavinson, D., McCarthy, J.E., Shapiro, H.S.: Best approximation in the mean by analytic and harmonic functions. Indiana Univ. Math. J. **49**(4), 1481–1513 (2000). URL https://doi.org/10.1512/iumj.2000.49.1787
11. Khavinson, D., Pérez-González, F., Shapiro, H.S.: Approximation in L^1-norm by elements of a uniform algebra. Constr. Approx. **14**(3), 401–410 (1998). URL https://doi.org/10.1007/s003659900080
12. Khavinson, S.Y.: Foundations of the theory of extremal problems for bounded analytic functions and various generalizations of them, *Amer. Math. Soc. Transl. (2)*, vol. 129, pp. 1–56 (1986)
13. Khavinson, S.Y.: Theory of factorization of single-valued analytic functions on compact Riemann surfaces with a boundary. Uspekhi Mat. Nauk **44**(4(268)), 155–189, 256 (1989). URL https://doi.org/10.1070/RM1989v044n04ABEH002146
14. Newman, D.J.: Pseudo-uniform convexity in H^1. Proc. Amer. Math. Soc. **14**, 676–679 (1963). URL https://doi.org/10.2307/2034299
15. Shapiro, H.S.: The Schwarz Function and its Generalization to Higher Dimensions, *University of Arkansas Lecture Notes in the Mathematical Sciences*, vol. 9. John Wiley & Sons, Inc., New York (1992). A Wiley-Interscience Publication
16. Totik, V.: Written communication (2019)